Symmetry, topology and the maximum number of mutually pairwise-touching infinite cylinders: configuration classification

Peter V. Pikhitsa and Stanislaw Pikhitsa

Article citation details
R. Soc. open sci. 4: 160729.
http://dx.doi.org/10.1098/rsos.160729

Review timeline
Original submission: 24 September 2016
Revised submission: 22 November 2016
Final acceptance: 21 December 2016

Note: Reports are unedited and appear as submitted by the referee. The review history appears in chronological order.

Note: This manuscript was transferred from another Royal Society journal with peer review.

Review History

RSOS-160729.R0 (Original submission)

Review form: Reviewer 1 (Adil Mughal)

Is the manuscript scientifically sound in its present form?
Yes

Are the interpretations and conclusions justified by the results?
Yes

Is the language acceptable?
Yes

Is it clear how to access all supporting data?
Yes

Do you have any ethical concerns with this paper?
No

Have you any concerns about statistical analyses in this paper?
No

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Recommendation?
Major revision is needed (please make suggestions in comments)

Comments to the Author(s)
The paper entitled “Symmetry, topology and the maximum…” by Pikhitsa and Pikhitsa is concerned with the classification of mutually touching pairwise infinite cylinders in 3D Euclidean space. As the paper stands I am sure it will be of interest to specialists in the very narrow field to whom the paper is addressed to. However, the paper is currently almost completely inaccessible to the general mathematical reader (in which I include scientists working in pure/applied mathematics – and - areas such as physics, chemistry and theoretical biology – all of which are areas in which this work could have an impact). Depending on the how widely the authors wish for their work to be received they may choose to adopt some (or all) of the recommendations below.

In the first case: It is not clear to me from the introduction if the problem that the authors are concerned with is: if at least one cylinder touches one other cylinder – or – if they are looking for configurations in which: each cylinder makes contact with every other cylinder. Some illustrative figures (toy examples – with only three or four cylinder) of the type presented in figure 4 would be useful here.

As a non-specialist I find the introduction far too terse. It would help the paper to be more accessible if the authors could provide the reader with some intuitive examples (i.e. toy models) with which to gain an initial understanding of the problem at hand.

Why are 7,8 and 9 mutually touching cylinders so interesting? (again I am not an expert – please try to help readers like me)

What is meant by a cylinder with arbitrary cross section? If a cylinder has a circular cross section – do the results in this paper extend to rods with an elliptical cross section? What about rods with a square cross section? Or a rectangular one? If it is rectangular I can make one side much longer than the other so that I have a 2D sheet – do your results still hold? If so – I don’t believe it! Please convince me…!


It’s very hard to follow what the meaning of (7), (8) and (9). Again if the introduction had included a toy example with the corresponding matrices worked out alongside it would be of use.

“Two cylinders are said to be in equal environment (EE) if two rows/columns that correspond to these two cylinders in the chirality matrix are identical or can be made identical by multiplying by -1.” – Why does multiplication by -1 achieve this? Can I have a picture showing the symmetries of the chirality matrix with an actual example alongside?

Continuing further: I’m afraid I’m beginning to lose my grasp (which was already weak) on what is going on. Although, the discussion on “cylinders of arbitrary cross-section” is useful. The discussion at the start of the section entitled “Collection of configurations of 7,8, and 9 mutually touching round infinite cylinders.” Is VERY dense and hard to follow.

The above points are just a representative example of the difficulties I have in accessing this paper. I’m sure I would not be alone in grasping the significance of this work. I hope the authors will take steps to improve the readability of the paper for non-specialists. I cannot comment on the validity of the findings since I am at present unable to fully comprehend the nature of the problem - and the solution - posed by the authors.
Review form: Reviewer 2

Is the manuscript scientifically sound in its present form?
Yes

Are the interpretations and conclusions justified by the results?
Yes

Is the language acceptable?
Yes

Is it clear how to access all supporting data?
Yes

Do you have any ethical concerns with this paper?
No

Have you any concerns about statistical analyses in this paper?
No

Recommendation?
Accept with minor revision (please list in comments)

Comments to the Author(s)
The authors address a topic that is mathematically interesting and their results are, in addition, useful to engineering (auxetic materials). I recommend that the manuscript be accepted. However there are a few places where the presentation is unclear and I suggest that these be improved:

1. The proofs of the Theorems on page 8: The authors are (presumably) using the result that any n+1-dimensional chirality matrix contains a (in fact n) n-dimensional chirality matrices. It would be helpful to state this. The program in Appendix 5 which is used to prove this result is hard to decipher - either a short description (in the Appendix) or some comments in the code would be helpful.

2. The reasoning behind the derivation of Q in Section 3 is not explained. Either the algorithm (lines 29-34) should be explained or a citation should be given.

3. The statement on lines 51-52 on page 12, that a89 'fits the impossible Penrose triangle' is intriguing but not clear. A verbal or annotated pictorial explanation would be helpful.

Decision letter (RSOS-160729)

04-Nov-2016

Dear Dr Pikhitsa,

The editors assigned to your paper (“Symmetry, topology and the maximum number of mutually pairwise touching infinite cylinders: configuration classification”) have now received comments from reviewers. We would like you to revise your paper in accordance with the referee and Subject Editor suggestions which can be found below (not including confidential reports to the Editor). Please note this decision does not guarantee eventual acceptance.
Please submit a copy of your revised paper within three weeks (i.e. by the 27-Nov-2016). If we do not hear from you within this time then it will be assumed that the paper has been withdrawn. In exceptional circumstances, extensions may be possible if agreed with the Editorial Office in advance. We do not allow multiple rounds of revision so we urge you to make every effort to fully address all of the comments at this stage. If deemed necessary by the Editors, your manuscript will be sent back to one or more of the original reviewers for assessment. If the original reviewers are not available we may invite new reviewers.

To revise your manuscript, log into http://mc.manuscriptcentral.com/rsos and enter your Author Centre, where you will find your manuscript title listed under "Manuscripts with Decisions." Under "Actions," click on "Create a Revision." Your manuscript number has been appended to denote a revision. Revise your manuscript and upload a new version through your Author Centre.

When submitting your revised manuscript, you must respond to the comments made by the referees and upload a file "Response to Referees" in "Section 6 - File Upload". Please use this to document how you have responded to the comments, and the adjustments you have made. In order to expedite the processing of the revised manuscript, please be as specific as possible in your response.

In addition to addressing all of the reviewers' and editor's comments please also ensure that your revised manuscript contains the following sections as appropriate before the reference list:

- **Ethics statement (if applicable)**
  If your study uses humans or animals please include details of the ethical approval received, including the name of the committee that granted approval. For human studies please also detail whether informed consent was obtained. For field studies on animals please include details of all permissions, licences and/or approvals granted to carry out the fieldwork.

- **Data accessibility**
  It is a condition of publication that all supporting data are made available either as supplementary information or preferably in a suitable permanent repository. The data accessibility section should state where the article's supporting data can be accessed. This section should also include details, where possible of where to access other relevant research materials such as statistical tools, protocols, software etc can be accessed. If the data have been deposited in an external repository this section should list the database, accession number and link to the DOI for all data from the article that have been made publicly available. Data sets that have been deposited in an external repository and have a DOI should also be appropriately cited in the manuscript and included in the reference list.

If you wish to submit your supporting data or code to Dryad (http://datadryad.org/), or modify your current submission to dryad, please use the following link: http://datadryad.org/submit?journalID=RSOS&manu=RSOS-160729

- **Competing interests**
  Please declare any financial or non-financial competing interests, or state that you have no competing interests.

- **Authors’ contributions**
  All submissions, other than those with a single author, must include an Authors’ Contributions section which individually lists the specific contribution of each author. The list of Authors should meet all of the following criteria; 1) substantial contributions to conception and design, or acquisition of data, or analysis and interpretation of data; 2) drafting the article or revising it critically for important intellectual content; and 3) final approval of the version to be published.
All contributors who do not meet all of these criteria should be included in the acknowledgements.

We suggest the following format:
AB carried out the molecular lab work, participated in data analysis, carried out sequence alignments, participated in the design of the study and drafted the manuscript; CD carried out the statistical analyses; EF collected field data; GH conceived of the study, designed the study, coordinated the study and helped draft the manuscript. All authors gave final approval for publication.

• Acknowledgements
Please acknowledge anyone who contributed to the study but did not meet the authorship criteria.

• Funding statement
Please list the source of funding for each author.

Once again, thank you for submitting your manuscript to Royal Society Open Science and I look forward to receiving your revision. If you have any questions at all, please do not hesitate to get in touch.

Yours sincerely,
Alice Power
Editorial Coordinator
Royal Society Open Science

on behalf of Mark Chaplain
Subject Editor, Royal Society Open Science
openscience@royalsociety.org

Comments to Author:

Reviewers' Comments to Author:
Reviewer: 1

Comments to the Author(s)
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In the first case: It is not clear to me from the introduction if the problem that the authors are concerned with is: if at least one cylinder touches one other cylinder – or – if they are looking for configurations in which: each cylinder makes contact with every other cylinder. Some illustrative figures (toy examples – with only three or four cylinder) of the type presented in figure 4 would be useful here.
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“Seidel adjacency matrix used in the theory of graphs” – what is the Seidel adjacency matrix?

It’s very hard to follow what the meaning of (7), (8) and (9). Again if the introduction had included a toy example with the corresponding matrices worked out alongside it would be of use.

“Two cylinders are said to be in equal environment (EE) if two rows/columns that correspond to these two cylinders in the chirality matrix are identical or can be made identical by multiplying by -1.” – Why does multiplication by -1 achieve this? Can I have a picture showing the symmetries of the chirality matrix with an actual example alongside?

Continuing further: I’m afraid I’m beginning to lose my grasp (which was already weak) on what is going on. Although, the discussion on “cylinders of arbitrary cross-section” is useful. The discussion at the start of the section entitled “Collection of configurations of 7,8, and 9 mutually touching round infinite cylinders.” Is VERY dense and hard to follow.

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Reviewer: 2

Comments to the Author(s)
The authors address a topic that is mathematically interesting and their results are, in addition, useful to engineering (auxetic materials). I recommend that the manuscript be accepted. However there are a few places where the presentation is unclear and I suggest that these be improved:

1. The proofs of the Theorems on page 8: The authors are (presumably) using the result that any n+1 - dimensional chirality matrix contains a (in fact n) n-dimensional chirality matrices. It would be helpful to state this. The program in Appendix 5 which is used to prove this result is hard to decipher - either a short description (in the Appendix) or some comments in the code would be helpful.

2. The reasoning behind the derivation of Q in Section 3 is not explained. Either the algorithm (lines 29-34) should be explained or a citation should be given.

3. The statement on lines 51-52 on page 12, that a89 'fits the impossible Penrose triangle' is intriguing but not clear. A verbal or annotated pictorial explanation would be helpful.
Author's Response to Decision Letter for (RSOS-160729)

See Appendix A.

RSOS-160729.R1 (Revision)

Review form: Reviewer 1 (Adil Mughal)

Is the manuscript scientifically sound in its present form?
Yes

Are the interpretations and conclusions justified by the results?
Yes

Is the language acceptable?
Yes

Is it clear how to access all supporting data?
Yes

Do you have any ethical concerns with this paper?
No

Have you any concerns about statistical analyses in this paper?
No

Recommendation?
Accept as is

Comments to the Author(s)
I am grateful to the authors for having made the necessary effort to make this article more assessable to the non-specialist. I feel they have done enough while presenting what remains a complex mathematical argument. I feel the article ought to be published in its revised form now.

Decision letter (RSOS-160729.R1)

21-Dec-2016

Dear Dr Pikhitsa,

I am pleased to inform you that your manuscript entitled "Symmetry, topology and the maximum number of mutually pairwise touching infinite cylinders: configuration classification" is now accepted for publication in Royal Society Open Science.

You can expect to receive a proof of your article in the near future. Please contact the editorial office (openscience_proofs@royalsociety.org and openscience@royalsociety.org) to let us know if you are likely to be away from e-mail contact. Due to rapid publication and an extremely tight schedule, if comments are not received, your paper may experience a delay in publication.
Royal Society Open Science operates under a continuous publication model (http://bit.ly/cpFAQ). Your article will be published straight into the next open issue and this will be the final version of the paper. As such, it can be cited immediately by other researchers. As the issue version of your paper will be the only version to be published I would advise you to check your proofs thoroughly as changes cannot be made once the paper is published.

In order to raise the profile of your paper once it is published, we can send through a PDF of your paper to selected colleagues. If you wish to take advantage of this, please reply to this email with the name and email addresses of up to 10 people who you feel would wish to read your article.

On behalf of the Editors of Royal Society Open Science, we look forward to your continued contributions to the Journal.

Kind regards,
Alice Power
Royal Society Open Science
openscience@royalsociety.org
http://rsos.royalsocietypublishing.org/

Reviewer(s)' Comments to Author:
Reviewer: 1

Comments to the Author(s)
I am grateful to the authors for having made the necessary effort to make this article more assessable to the non-specialist. I feel they have done enough while presenting what remains a complex mathematical argument. I feel the article ought to be published in its revised form now.
Replies to the Reviewers

We revised our MS carefully and accordingly to the Reviewers comments. Our answers to the Reviewers and all changes made in the revised MS are marked in red.

Comments to Author:

Reviewers’ Comments to Author:
Reviewer: 1

Comments to the Author(s)
The paper entitled “Symmetry, topology and the maximum…” by Pikhitsa and Pikhitsa is concerned with the classification of mutually touching pairwise infinite cylinders in 3D Euclidean space. As the paper stands I am sure it will be of interest to specialists in the very narrow field to whom the paper is addressed to. However, the paper is currently almost completely inaccessible to the general mathematical reader (in which I include scientists working in pure/applied mathematics – and - areas such as physics, chemistry and theoretical biology – all of which are areas in which this work could have an impact). Depending on the how widely the authors wish for their work to be received they may choose to adopt some (or all) of the recommendations below.

We would like to thank the Reviewer for the advice to improve the readability of the article which we have gladly followed.

In the first case: It is not clear to me from the introduction if the problem that the authors are concerned with is: if at least one cylinder touches one other cylinder – or – if they are looking for configurations in which: each cylinder makes contact with every other cylinder. Some illustrative figures (toy examples – with only three or four cylinder) of the type presented in figure 4 would be useful here.

As a non-specialist I find the introduction far too terse. It would help the paper to be more accessible if the authors could provide the reader with some intuitive examples (i.e. toy models) with which to gain an initial understanding of the problem at hand.

In the introduction we used the words “mutually touching” which assumes that each cylinder makes contact with every other cylinder. We rewrote the introduction in the revised MS now explaining the simple geometry of the problem by describing the process of stepwise creating configuration examples and by making reference to Fig.1b of our paper [2] in pssb 2009 where the configuration of the mutually touching 7 round equal cylinders was published for the first time (Fig. R1).
We put into the revised MS a paragraph:
“In order to construct a configuration of mutually touching cylinders one may begin with two infinite round cylinders in a mutual touch at one point (like scissors) and then add the third one so that it simultaneously touches both cylinders. Then one adds the fourth cylinder to touch simultaneously each of the rest ones. Beginning from the fifth cylinder further adding the cylinders grows harder. Thus the problem of how many infinite round cylinders in 3D it is possible to put into the mutual touching arises.”

Why are 7, 8 and 9 mutually touching cylinders so interesting? (again I am not an expert – please try to help readers like me)

7 is the maximum number of mutually touching equal round cylinders which we found in [2]. In case of equal spheres this number is 4. These numbers are fundamental and reflect the basic properties of packing in 3D space. But unlike spheres, cylinders are additionally characterized by orientations (angular variables) which become entangled with the spatial variables through the coordinates of the contact points of the cylinders [1]. The importance of such a problem lies in a challenge for modern mathematics which has difficulties with solving entangled configurations. For example, the complexity of the knot theory is well known. Yet the entanglement of many cylinders may look even more complicated because of the multicomponent nature of the problem. If one allows the cylinders to have more freedom (the first step was to allow round cylinders to have arbitrary radii) then (we showed that) the number of the mutually touching cylinders grows to maximum 9.

What is meant by a cylinder with arbitrary cross section? If a cylinder has a circular cross section – do the results in this paper extend to rods with an elliptical cross section? What about rods with
a square cross section? Or a rectangular one? If it is rectangular I can make one side much longer than the other so that I have a 2D sheet – do your results still hold? If so – I don’t believe it! Please convince me…!

As we wrote on page 8 “Still it is possible to consider infinite straight stripes (blades) with straight line edges if only the pairwise contact is allowed which we will publish elsewhere.” Certainly our results extend to rods with an elliptical cross section and hold even for 2D flat infinite stripes/sheets being a degenerate case of the cylinders. The only important restriction for the general result of the cylinders of arbitrary cross-section to hold for infinitely thin stripes/blades is to forbid more than two infinitely thin blades touching in one single point (multiple touching may be possible due to their degeneracy in thickness). We easily calculated a 7 stripe configuration of equal stripes as an example (see Fig. R2) and we plan to consider this special degenerate case in full and publish elsewhere.

Fig. R2.

“Seidel adjacency matrix used in the theory of graphs” – what is the Seidel adjacency matrix?

The Seidel adjacency matrix is a special matrix from the Theory of Graphs. Its definition is: the Seidel adjacency matrix of a simple undirected graph is a symmetric matrix with a row and column for each vertex, having 0 on the diagonal, −1 for positions whose rows and columns correspond to adjacent vertices, and +1 for positions corresponding to non-adjacent vertices. We included this definition in the revised MS.

“According to its definition, the Seidel adjacency matrix of a simple undirected graph is a symmetric matrix with a row and column for each vertex, having 0 on the diagonal, −1 for positions whose rows and columns correspond to adjacent vertices, and +1 for positions corresponding to non-adjacent vertices.”

It’s very hard to follow what the meaning of (7), (8) and (9). Again if the introduction had included a toy example with the corresponding matrices worked out alongside it would be of use.

Unfortunately, it is not easy to simplify the situation without losing its natural complexity because any configuration that makes sense is multicomponent: even to make a ring that
encages/contains one cylinder already takes at least four cylinders. We do believe that following our recommendation given in the text the ring matrices (7), (8) and chirality matrix (9) can be understood after some efforts:

“It is remarkable that one can reproduce the chirality matrix and the ring matrix just by inspecting any given configuration while establishing directions along the cylinders and marking the chirality at each contact, along with the counting the number of rings around each cylinder that contain this cylinder.”

“Two cylinders are said to be in equal environment (EE) if two rows/columns that correspond to these two cylinders in the chirality matrix are identical or can be made identical by multiplying by -1.” – Why does multiplication by -1 achieve this? Can I have a picture showing the symmetries of the chirality matrix with an actual example alongside?

The answer is simple: multiplying the row/column by -1 just changes the orientation of the corresponding cylinder to the opposite direction which is not relevant for the EE configuration because the cylinder is symmetric and should be characterized rather by a director then by a vector. We added in the revised MS:

“if the direction of a line is switched to the opposite one then the corresponding matrix entries are multiplied by -1 and the chirality matrix becomes different. On the other hand, any numerical invariant based on the chirality matrix should not depend on switching line direction to opposite (because cylinders are symmetric) and thus one has to construct a matrix that is degenerate towards opposite line directions.”

Continuing further: I’m afraid I’m beginning to lose my grasp (which was already weak) on what is going on. Although, the discussion on “cylinders of arbitrary cross-section” is useful. The discussion at the start of the section entitled “Collection of configurations of 7,8, and 9 mutually touching round infinite cylinders.” Is VERY dense and hard to follow.

We should confess that it is true and the section should be difficult to follow. Yet the section has only an illustrative character and being considered superficially is just a list of configurations but it may be useful for a deeply interested reader.

The above points are just a representative example of the difficulties I have in accessing this paper. I’m sure I would not be alone in grasping the significance of this work. I hope the authors will take steps to improve the readability of the paper for non-specialists. I cannot comment on the validity of the findings since I am at present unable to fully comprehend the nature of the problem - and the solution - posed by the authors.

Once again we thank Reviewer 1 for valuable comments. We hope that the revised MS may have become easier to read and comprehend.

Reviewer: 2

Comments to the Author(s)
The authors address a topic that is mathematically interesting and their results are, in addition, useful to engineering (auxetic materials). I recommend that the manuscript be accepted. However there are a few places where the presentation is unclear and I suggest that these be improved:

We thank Reviewer 2 for his favorable attitude and valuable comments.

1. The proofs of the Theorems on page 8: The authors are (presumably) using the result that any $n+1$ - dimensional chirality matrix contains a (in fact $n$) n-dimensional chirality matrices. It would be helpful to state this.

We have stated this in the revised MS:
“(using the fact that any $n + 1$ - dimensional chirality matrix contains $n$ submatrices being $n$-dimensional chirality matrices)”.

The program in Appendix 5 which is used to prove this result is hard to decipher - either a short description (in the Appendix) or some comments in the code would be helpful.

We have given a short description of the code algorithm in the text of the revised MS:
“Briefly, the code algorithm uses recursive/iterative exhaustive search and is as follows. First, we look up through all possible $5 \times 5$ symmetric matrices with the zero diagonal and all possible combinations of entries $\pm 1$. Keep those matrices that have a “unique” set of eigenvalues which means that if during sorting one comes across a matrix with the same set of eigenvalues as the set of a previously found matrix of the unique set, such a matrix is ignored as not being unique. Those matrices that have the set of eigenvalues which coincides with the set of $K5$ are also ignored. Second, we add an additional row (along with the transposed column to keep the matrix symmetric) with one zero and all other possible entries $\pm 1$ to each of the unique matrices of the first step to obtain a set of $6 \times 6$ symmetric matrices. While trying all possible entries $\pm 1$ for the attached row/column for each of the matrices we select a “unique” set of $6 \times 6$ symmetric matrices. Then of the selected “unique” ones we sort out those that have $5 \times 5$ symmetric submatrices with the same set of eigenvalues as $K5$ has. Third, the procedure of adding a row/column to the unique matrices and the sorting are repeated again and again until there is no matrix left in the set. As is seen, the largest matrix that can be obtained in this way is the $18 \times 18$ matrix given above.”

2. The reasoning behind the derivation of $Q$ in Section 3 is not explained. Either the algorithm (lines 29-34) should be explained or a citation should be given.

We have explained the derivation of $Q$ in Section 3 in the revised MS by adding the following:
“Unfortunately, the chirality matrix is sensitive to the opposite orientations of the lines: if the direction of a line is switched to the opposite one then the corresponding matrix entries are multiplied by -1 and the chirality matrix becomes different. On the other hand, any numerical invariant based on the chirality matrix should not depend on switching line direction to opposite (because cylinders are symmetric) and thus one has to construct a matrix that is degenerate towards opposite line directions. Therefore we introduce a new matrix $Q$ derived from $P$ which is invariant with respect to the line orientation as follows (we are using Eq. (9) as an example):
One has to transform the first row in $P(a89)$ into all +1s (it is already so) and to sum up numbers in the corresponding columns and put the sums into the first row.

One has to transform the second row in $P(a89)$ into all +1s (here by reverting sign in the sixth column and then in the sixth row) then to sum up numbers in the corresponding columns and put the sums into the second row.

After proceeding in the same way through the whole matrix of Eq. (9) one gets the symmetric matrix:

3. The statement on lines 51-52 on page 12, that a89 'fits the impossible Penrose triangle' is intriguing but not clear. A verbal or annotated pictorial explanation would be helpful.

Additionally to Fig. 11, in the revised MS we give a verbal explanation:
“Indeed, one can uniformly decorate the three “corner” cubes of the impossible Penrose triangle with balls as it is shown in Fig. 11. After coloration, the balls on the corner cubes determine the position of the lines of the corresponding colors that pass through them. The configuration a89 is reproduced when three pairs of lines (brown-green, cyan-blue, and violet-grey) in Fig. 11 are aligned along the three sides of the triangle so that while going around the central hole (which should be filled with the red pivot cylinder to complete a89) and starting from the brown-green pair one can see the right-hand screw of the pair, then the left-hand screw, and finally, the left-hand screw of the violet-grey pair of lines.”